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# Spin structures on real projective quadrics

Michel Cahen
*Département de Mathématiques, Université Libre de Bruxelles, C.P. 218, Campus Plaine, B-1050 Brussels, Belgium*

Simone Gutt
*Département de Mathématiques, Université de Metz, Ile du Saulcy, F-57045 Metz Cedex 01, France*

Andrzej Trautman [1]
*Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoza 69, PL-00681 Warsaw, Poland*

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Projective quadrics are known to be conformal compactifications of Euclidean spaces. In particular, the (projective) real quadric $Q_{p,q} = (S_p \times S_q)/\mathbb{Z}_2$ is associated, in this manner, with the flat space $\mathbb{R}^{p+q}$ endowed with a metric tensor of signature $(p,q)$. For $p$ and $q$ positive, the quadric $Q_{p,q}$ is orientable iff $p+q$ is even. The quadric has two natural metrics, invariant with respect to the action of $\mathrm{O}(p+1) \times \mathrm{O}(q+1)$: a proper Riemannian one and a pseudo-Riemannian metric of signature $(p,q)$. This paper contains an explicit description of spin structures on real, even-dimensional quadrics for both metrics, whenever these structures exist. In particular, it is shown that, for $p$ and $q$ even positive, the proper (pseudo-Riemannian) metric gives rise to two inequivalent spin structures iff $p+q \equiv 2 \pmod 4$ ($p+q \equiv 0 \pmod 4$). If $p$ and $q$ are odd and $> 1$, then there is no spin structure for either metric whenever $p+q \equiv 0 \pmod 4$; otherwise, there are two spin structures for each of the metrics. There always exist spin structures on real quadrics with a Lorentzian metric, i.e., when $p$ and $q$ are odd and $p$ or $q = 1$.

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*Dedicated to Roger Penrose*

[1] Present address: Interdisciplinary Laboratory for Natural and Humanistic Sciences of the International School for Advanced Studies, 34014 Trieste, Italy.

## 1. Introduction

Roger Penrose proposed to use the conformal compactification $M$ of Minkowski space–time for a convenient description of fields and particles of zero mass [1]. The manifold $M$ is diffeomorphic to the real projective quadric $(S_1 \times S_3)/\mathbb{Z}_2$. The canonical Riemannian metrics on the circle $S_1$ and the three-sphere $S_3$ define, in a natural manner, two metrics on $M$: a proper Riemannian one and a pseudo-Riemannian metric of signature (1,3). The manifold $M$ with either of the two metrics admits two inequivalent spin structures. Penrose and Rindler [2] gave a description of these two structures—for the pseudo-Riemannian case—in a geometrical language adapted to the needs of physics; see also ref. [3].

In this paper, we take up the general problem of finding the spin structures on the proper $(p, q > 0)$ real (projective) quadrics

$$Q_{p,q} = (S_p \times S_q)/\mathbb{Z}_2$$

endowed with one of the two natural metrics (proper or pseudo-Riemannian) descending from $S_p \times S_q$. (To alleviate the language, from now on, the adjective "projective" will be omitted from the expression "projective quadric".) We restrict ourselves to even-dimensional quadrics, because, as we show in section 4, the manifold $Q_{p,q}$ is orientable if, and only if, $p + q$ is even. The study of the odd-dimensional case requires the consideration of pin structures. Our work prepares ground for the computation of the spectrum of the Dirac operator on $Q_{p,q}$.

The manifold $Q_{p,q}$ can be regarded as a "real form" of the complex quadric $Q_n$ of complex dimension $n = p + q$. In fact, the embedding of $Q_{1,3}$ in $Q_4$, the Grassmannian of complex two-planes in $\mathbb{C}^4$, plays a fundamental role in twistor theory [2,4–6]. The quadrics $Q_n$ have a conformal structure, but no complex bilinear Riemannian metric. For this reason, it is appropriate to define on $Q_n$ a *conformal spin structure* as a prolongation of its bundle of conformal frames corresponding to the non-trivial extension of the complex conformal group by $\mathbb{Z}_2$ [7]. It is interesting to relate the spin structures on $Q_{p,q}$ to the conformal spin structures on $Q_{p+q}$ [8].

The nature of the construction of spin structures we present here is differential-geometric and Lie group-theoretic. Our method is an extension of the one used in the study of spin (and pin) structures on spheres and projective spaces [9,10] and simply connected Riemannian symmetric spaces [11]. The results concerning existence and the number of inequivalent spin structures on $Q_{p,q}$, $p + q = 2m$, where $p, q$ and $m$ are positive integers, are summarized in table 1.

In particular, for the familiar real forms of $Q_4$ we have the following: as is well known, the sphere $S_4 = Q_{0,4}$ admits one spin structure; the "neutral form" $Q_{2,2}$ has two spin structures, but the proper Riemannian metric on $(S_2 \times S_2)/\mathbb{Z}_2$

Table 1

Summary of results concerning existence and the number of inequivalent spin structures on $Q_{p,q}$, $p + q = 2m$, where $p, q$ and $m$ are positive integers. Here the words "proper" and "pseudo" refer to the natural, proper Riemannian and pseudo-Riemannian metrics, respectively, and the figures in the last two columns indicate the numbers of inequivalent spin structures.

| $p$ and $q > 0$ | $m$ mod 2 | proper | pseudo |
|---|---|---|---|
| even | 0 | 0 | 2 |
| | 1 | 2 | 0 |
| odd, $p, q > 1$ | 0 | 0 | 0 |
| | 1 | 2 | 2 |
| $p = 1, q$ odd $> 1$ | 0 or 1 | 2 | 2 |
| $p = q = 1$ | 1 | 4 | 4 |

does not give rise to any spin structure; the compactified Minkowski space has two spin structures irrespective of whether it is given a Lorentzian or a proper Riemannian metric. The quadric $Q_{3,5}$ has no spin structure for either of the two metrics.

The paper is organized as follows: The next section contains a summary of our notation and terminology. Section 3 contains a brief review of the conformal geometry of projective quadrics. We give, in particular, a short description of the conformal embedding of a pseudo-Euclidean space into an appropriate quadric and of the associated action of the isotropy group of a "point at infinity"; this is well known, but hard to find in print. In section 4, we prove that real quadrics are symmetric spaces and exhibit their groups of isometries. The following section contains theorems on spin structures of homogeneous Riemannian manifolds and of products of spin manifolds. These theorems serve as tools to construct spin structures on the real quadrics (section 6). The last part of the paper contains a brief comparison of our results with those that can be inferred, on the existence of spin structures, from the computation of the Stiefel–Whitney classes of the real quadrics.

## 2. Notation and preliminaries

Our notation and terminology follow the custom prevaling in differential geometry and mathematical physics; we often use refs. [12–14]. In this section, we summarize some of our notation and recall the definitions and properties of Clifford algebras and spin groups relevant to our work on spin structures; further details may be found in refs. [7, 15–18].

## 2.1. DIFFERENTIAL GEOMETRY

We work in the category of finite-dimensional, smooth (i.e. of class $C^\infty$) manifolds and often omit the adjective "smooth". If $M$ is such a manifold, then $TM$ is (the total space of) its tangent bundle; if $f : M \to N$ is smooth, then $f_* : TM \to TN$ is its tangent (derived) map and the symbol $f^*$ is used to denote the corresponding pullback of differential forms from $N$ to $M$. If $E \to M$ is a vector bundle over $M$, then its fibre $E_x$ over $x \in M$ is a vector space. If $F \to M$ is another vector bundle, then there are the bundles $E \oplus F \to M$ (the Whitney sum) and $\mathrm{Hom}(E,F) \to M$ such that $(E \oplus F)_x = E_x \oplus F_x$ and $\mathrm{Hom}(E,F)_x = \mathrm{Hom}(E_x, F_x)$. If $F \to M$ is a vector subbundle of the vector bundle $E \to M$, then there is the quotient bundle $E/F \to M$ etc.

The fibre of $TM \to M$ is the tangent space $T_xM$ to $M$ at $x$. A frame (linear basis) at $x$ is an isomorphism of vector spaces $\xi : \mathbb{R}^n \to T_xM$, where $n = \dim M$. If $a \in \mathrm{GL}(n,\mathbb{R})$, then the composition $\xi a$ is also a frame at $x$. The set of all frames on $M$ is made into the total space of a principal $\mathrm{GL}(n,\mathbb{R})$-bundle, the bundle of linear frames.

If $P \to M$ is a principal $G$-bundle, then the Lie group $G$ acts on $P$ to the right; the action map is $P \times G \ni (\xi, a) \to \xi a \in P$. Let $\rho : H \to G$ be a homomorphism of Lie groups; a map $\sigma : Q \to P$ is a morphism of the $H$-bundle $Q \to M$ into the $G$-bundle $P \to M$, corresponding to $\rho$, if the diagram

$$\begin{array}{ccc} Q \times H & \xrightarrow{\sigma \times \rho} & P \times G \\ \downarrow & & \downarrow \\ Q & \xrightarrow{\sigma} & P \\ & \searrow \quad \swarrow & \\ & M & \end{array} \tag{2.1}$$

is commutative. In particular, if $\rho$ and $\sigma$ are surjective (injective), then $\sigma : Q \to P$ is said to be a *prolongation* (*restriction*) of $P$ to $H$. There are topological obstructions to the existence of prolongations and restrictions. On the other hand, given an $H$-bundle $Q \to M$ and a homomorphism $\rho : H \to G$ there is always an extension of $Q$ to $G$, i.e. a $G$-bundle $P \to M$ and a morphism $\sigma : Q \to P$, corresponding to $\rho$. The extension is a bundle *associated* with the $H$-bundle $Q \to M$; the total space $P$ is $Q \times_\rho G$, the set of equivalence classes $[(\eta, a)]$, where $\eta \in Q$, $a \in G$ and $[(\eta, a)] = [(\eta', a')]$ if, and only if, there exists $b \in H$ such that $\eta' = \eta b$ and $a = \rho(b)a'$.

If $H = G$, $\rho = \mathrm{id}$ and $\sigma$ is a diffeomorphism, then $\sigma$ is said to be an *isomorphism* of $G$-bundles over $M$. Consider now two prolongations $\sigma_i : Q_i \to P$ ($i = 1, 2$) of the $G$-bundle $P$ to $H$, both corresponding to $\rho : H \to G$. These prolongations are said to be *equivalent* if there is an isomorphism $\kappa : Q_1 \to Q_2$ of $H$-bundles over $M$ such that $\sigma_2 \circ \kappa = \sigma_1$.

If $M$ is an oriented, proper Riemannian $n$-manifold, then its bundle of linear frames restricts to $\mathrm{SO}(n)$; similarly, giving on $M$ an orientation and a metric tensor of signature $(p,q)$ is equivalent to restricting its bundle of linear frames to $\mathrm{SO}(p,q)$. If the latter bundle can be further restricted to $\mathrm{SO}^0(p,q)$, the connected open subgroup of $\mathrm{SO}(p,q)$, then $M$ is said to be *space-* and *time-orientable.*

### 2.2. CLIFFORD ALGEBRAS AND SPIN GROUPS

An algebra $A$ over $K = \mathbb{R}$ or $\mathbb{C}$ is said to be $\mathbb{Z}_2$*-graded* if there is a decomposition $A = A_0 \oplus A_1$ such that $A_i A_j \subset A_{i+j}$, where $i+j$ is understood mod 2. If $a \in A_i$, then $i = \varepsilon(a)$ is the degree of $a$. Elements of degree 0 (degree 1) are called even (odd) and $A_0$ is the *even subalgebra* of $A$. A $\mathbb{Z}_2$-grading of $A$ is equivalent to giving an involution $\alpha$ of $A$ characterized by the property $\alpha(a) = (-1)^{\varepsilon(a)}a$ for $a$ homogeneous (i.e. of definite degree). Given a $\mathbb{Z}_2$-graded algebra $A$, one defines the *twisted* algebra $A^{\mathrm{T}}$ as having the same underlying vector space and grading as $A$, whereas the product of two homogeneous elements $a$ and $b$ is given in $A^{\mathrm{T}}$ by

$$a \top b = (-1)^{\varepsilon(a)\varepsilon(b)} ab.$$

The even subalgebras of $A$ and $A^{\mathrm{T}}$ are isomorphic.

Let $V$ be a finite-dimensional vector space over $K = \mathbb{R}$ or $\mathbb{C}$ with a scalar product $g$, defined to be a bilinear, symmetric and non-degenerate map $g : V \times V \to K$. The Clifford algebra $\mathrm{Cl}(g)$ is the universal associative algebra over $K$ with unit, generated by $K \oplus V$, subject to all relations of the form

$$uv + vu = 2g(u,v), \quad u,v \in V.$$

Universality implies that, if $A$ is another associative algebra over $K$ with unit and $f : V \to A$ is a *Clifford map for* $g$, i.e. a linear map such that $f(v)^2 = g(v,v)$ for every $v \in V$, then there is a homomorphism of algebras with unit $\mathrm{Cl}(g) \to A$, extending the map $f$. In particular, the Clifford map for $g$,

$$V \to \mathrm{Cl}(g), \quad v \mapsto -v$$

extends to the involution $\alpha$ of $\mathrm{Cl}(g)$, defining its $\mathbb{Z}_2$-grading. The even subalgebra of $\mathrm{Cl}(g)$ is denoted by $\mathrm{Cl}_0(g)$. In general, the algebras $\mathrm{Cl}(g)$ and $\mathrm{Cl}(-g)$ are not isomorphic. However, the canonical injection $V \to \mathrm{Cl}(-g)^{\mathrm{T}}$ is a Clifford map for $g$ and extends to the isomorphism of algebras with unit

$$\imath : \mathrm{Cl}(g) \to \mathrm{Cl}(-g)^{\mathrm{T}}$$

such that $\imath(v)^2 = g(v,v)$. Restricted to $\mathrm{Cl}_0(g)$, this gives a natural isomorphism

$$\imath : \mathrm{Cl}_0(g) \to \mathrm{Cl}_0(-g). \tag{2.2}$$

Let $V$ and $W$ be vector spaces over $K$ with scalar products $g$ and $h$, respectively. Their direct sum $V \oplus W$ has an obvious scalar product $g \oplus h$. The injections $V \to V \oplus W$ and $W \to V \oplus W$ extend to monomorphisms of algebras $\mathrm{Cl}(g) \to \mathrm{Cl}(g \oplus h)$ and $\mathrm{Cl}(h) \to \mathrm{Cl}(g \oplus h)$; the algebras $\mathrm{Cl}(g)$ and $\mathrm{Cl}(h)$ are identified with their images in $\mathrm{Cl}(g \oplus h)$. The map

$$\mathrm{Cl}_0(g) \otimes \mathrm{Cl}_0(h) \to \mathrm{Cl}_0(g \oplus h), \tag{2.3}$$

defined by $a \otimes b \mapsto ab$, is a homomorphism of algebras.

The "numeric" vector space $K^n$ ($K = \mathbb{R}$ or $\mathbb{C}$) has the *standard* quadratic form

$$(z|z) = z_1^2 + \cdots + z_n^2, \quad z = (z_1, \dots, z_n) \in K^n \tag{2.4}$$

and there is the associated scalar product $(z|w)$ of vectors $z, w \in K^n$.

From now on, through to the end of this section, we consider vector spaces and algebras over the reals only. We say that $u \in V$ is a unit vector if $g(u,u) = u^2 = 1$ or $-1$ and note that the map $v \mapsto -uvu^{-1}, v \in V$, is a reflection in the hyperplane orthogonal to $u$. The *spin group* $\mathrm{Spin}(g) \subset \mathrm{Cl}_0(g)$ is defined as the set of Clifford products of all even sequences of unit vectors; the group multiplication is induced by the Clifford product. If $a \in \mathrm{Spin}(g)$ and $v \in V$, then $\rho(a) = ava^{-1}$ is a vector with the same square as $v$. Since $\rho(a)$ is the composition of an even sequence of reflections in hyperplanes, it is a proper orthogonal transformation. By the Cartan–Dieudonné theorem (ref. [19], §6, prop. 5), every such transformation can be represented as the composition of an even sequence of reflections so that $\rho$ is a homomorphism of $\mathrm{Spin}(g)$ onto $\mathrm{SO}(g)$, the group of orientation-preserving automorphisms of $V$, orthogonal with respect to $g$. One shows that $\ker \rho = \{1, -1\} = \mathbb{Z}_2$; there is the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(g) \to \mathrm{SO}(g) \to 1.$$

By restriction, one obtains from (2.2) and (2.3) the isomorphism of groups

$$\imath : \mathrm{Spin}(g) \to \mathrm{Spin}(-g), \tag{2.5}$$

and the homomorphism

$$\mathrm{Spin}(g) \times \mathrm{Spin}(h) \to \mathrm{Spin}(g \oplus h), \quad (a,b) \mapsto ab. \tag{2.6}$$

By combining (2.5) and (2.6) one obtains also the homorphism

$$\mathrm{Spin}(g) \times \mathrm{Spin}(h) \to \mathrm{Spin}(g \oplus (-h)), \quad (a,b) \mapsto a\imath(b). \tag{2.7}$$

If $V = \mathbb{R}^p \times \mathbb{R}^q$ and $g$ is of signature $(p,q)$,

$$g(u,u) = (x|x) - (y|y), \quad u = (x,y) \in \mathbb{R}^p \times \mathbb{R}^q, \tag{2.8}$$

then we write $\mathrm{Cl}(p,q)$, $\mathrm{Spin}(p,q)$, etc., instead of $\mathrm{Cl}(g)$, $\mathrm{Spin}(g)$, etc.

The groups $\mathrm{Spin}(n) = \mathrm{Spin}(n,0)$, $n = 1, 2, \dots$, are compact; they are connected for $n \geq 2$ and $\mathrm{Spin}(1) = \mathbb{Z}_2$. For $pq \neq 0$, the group $\mathrm{Spin}(p,q)$ is non-

Table 2
The fundamental groups of $\mathrm{Spin}^0(p,q)$.

| $p$ and $q$ | $\pi_1(\mathrm{Spin}^0(p,q))$ |
|---|---|
| $p = 0$ or $1, q \neq 2$ | 0 |
| $p = q = 2$ | $\mathbb{Z} \oplus \mathbb{Z}$ |
| $p = 2, q \neq 2$ | $\mathbb{Z}$ |
| $p$ and $q \geq 3$ | $\mathbb{Z}_2$ |

compact and has two connected components. The component of the unit element, $\mathrm{Spin}^0(p,q)$, gives rise to the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}^0(p,q) \to \mathrm{SO}^0(p,q) \to 1.$$

The group $\mathrm{Spin}^0(p,q)$ can be equivalently defined as consisting of products of all sequences containing an even number of space-like and an even number of time-like unit vectors.

With a suitable specification of $g$ and $h$, the homomorphisms (2.6) and (2.7) reduce to the homomorphisms

$$\varphi_+ : \mathrm{Spin}(p) \times \mathrm{Spin}(q) \to \mathrm{Spin}(p+q), \tag{2.9}$$

$$\varphi_- : \mathrm{Spin}(p) \times \mathrm{Spin}(q) \to \mathrm{Spin}^0(p,q). \tag{2.10}$$

The fundamental groups of $\mathrm{Spin}^0(p,q)$ are given in table 2.

The group $\mathrm{Spin}(p,q)$ has a natural structure of a Lie group. Its Lie algebra can be identified with the vector subspace of $\mathrm{Cl}_0(p,q)$ spanned by all elements of the form $uv$, where $(u,v)$ is a pair of orthogonal unit vectors. If $u^2 = v^2$, then $\rho(\cos t + uv \sin t)$ is the rotation by the angle $2t$ in the plane spanned by $u$ and $v$. If $u^2 = -v^2$, then $\rho(\cosh t + uv \sinh t)$ is a special Lorentz transformation for every $t \in \mathbb{R}$ and $uv$ is an element of $\mathrm{Spin}(p,q)$ not in $\mathrm{Spin}^0(p,q)$.

Let $(e_1, \cdots, e_p, f_1, \cdots, f_q)$ be the standard linear basis in $\mathbb{R}^p \times \mathbb{R}^q$ orthonormal with respect to (2.8). If $p+q$ is even, $p+q=2m$, then the volume element

$$\mathrm{vol}_{p,q} = e_1 \cdots e_p f_1 \cdots f_q \tag{2.11}$$

is in the centre of $\mathrm{Spin}(p,q)$; it is in $\mathrm{Spin}^0(p,q)$ if, and only if, both $p$ and $q$ are even. Its square is

$$\mathrm{vol}_{p,q}^2 = (-1)^{m+p}. \tag{2.12}$$

We write

$$\mathrm{vol}_{2m} = \mathrm{vol}_{2m,0} \tag{2.13}$$

and note

$$\mathrm{vol}_{2m}^2 = (-1)^m. \tag{2.14}$$

## 3. Quadrics and conformal compactification

### 3.1. PROJECTIVE SPACES

Consider a $(n+2)$-dimensional vector space $W$ over $K = \mathbb{R}$ or $\mathbb{C}$. The set $W' \subset W$ of all non-zero vectors is an open (real or complex) submanifold of $W$ and the equivalence relation $R$ in $W'$,

$$w_1 \equiv w_2 \bmod R \text{ iff there is } \lambda \in K \text{ such that } w_2 = \lambda w_1, \tag{3.1}$$

is regular: the *projective space* $\mathbb{P} = W'/R$ has a natural topology and a differential structure that make it into a compact manifold of dimension $n+1$ over $K$ such that the canonical projection $W' \to \mathbb{P}$ is a submersion. For every $w \in W'$ there is the line through $w$,

$$[w] = \{\lambda w \in W : \lambda \in K\} \tag{3.2}$$

and $\mathbb{P}$ can be identified with the set of all such lines. The manifold $TW'$ can be identified with $W' \times W$: if $u : \mathbb{R} \to W'$ is a curve, then its tangent vector at $t \in \mathbb{R}$ is characterized by $(u(t), \dot{u}(t)) \in W' \times W$. The equivalence relation (3.1) extends to $TW' = W' \times W$,

$$\begin{aligned}(w_1, v_1) \equiv (w_2, v_2) \bmod R \text{ iff there are } \lambda, \mu \in K \\ \text{such that } w_2 = \lambda w_1 \text{ and } v_2 = \lambda v_1 + \mu w_1,\end{aligned} \tag{3.3}$$

and $T\mathbb{P}$ is then identified with $TW'/R$. If $(w,v) \in W' \times W$, then $[(w,v)]$ denotes its class with respect to the equivalence relation $R$ defined by (3.3); this is a vector tangent to $\mathbb{P}$ at $[w]$.

Let $E = \mathbb{P} \times W$ be the total space of the trivial bundle $E \to \mathbb{P}$ and

$$F = \{([w], v) \in E : v \in [w]\}$$

that of the *canonical line bundle* of $\mathbb{P}$. Note that the fibre of $F$ over $[w] \in \mathbb{P}$ is the line $[w]$ itself. With the vector $[(w,v)] \in T\mathbb{P}$ one associates the linear map $[w] \to W/[w]$ such that $w \to v + [w]$. This observation leads to the proof of a natural isomorphism of vector bundles over $\mathbb{P}$,

$$T\mathbb{P} \cong \mathrm{Hom}(F, E/F). \tag{3.4}$$

The general linear group $\mathrm{GL}(W)$ acts transitively on $W'$ and, by

$$a[w] = [aw], \quad a \in \mathrm{GL}(W),$$

also on $\mathbb{P}$. This action lifts to $T\mathbb{P}$,

$$a_*[(w,v)] = [(aw, av)]. \tag{3.5}$$

### 3.2. QUADRICS

Assume now that $W$ is given a scalar product $g$ such that there exist two vectors $w_0$ and $w_\infty$ with the property

$$g(w_0, w_0) = g(w_\infty, w_\infty) = 0, \qquad g(w_0, w_\infty) = 1/2. \tag{3.6}$$

For $K = \mathbb{C}$ this assumption is equivalent to $n \geq 0$; for $K = \mathbb{R}$ it is equivalent to the statement that the signature of $g$ is $(p+1, q+1)$ with $p$ and $q \geq 0$. *From now we assume* $n \geq 1$.

The *light cone*,

$$N = \{w \in W' : g(w,w) = 0\},$$

is an $(n+1)$-dimensional submanifold (hypersurface) in $W'$ and the *quadric*,

$$Q = \{[w] \in \mathbb{P} : w \in N\},$$

is an $n$-dimensional submanifold (hypersurface) in $\mathbb{P}$. The map $W' \to \mathbb{P}$ restricts to a submersion $N \to Q$. By considering the tangent to a curve in $N$ one obtains

$$TN = \{(w,v) \in N \times W : g(w,v) = 0\}$$

and also

$$TQ = \{[(w,v)] \in T\mathbb{P} : (w,v) \in TN\}. \tag{3.7}$$

The vector bundle $E \to \mathbb{P}$ restricts to a vector bundle over $Q$; similarly the subbundle $F \to \mathbb{P}$ restricts to a line subbundle over $Q$; we use the same letter $E$ $(F)$ to denote the total space of this induced bundle. The special orthogonal group $\mathrm{SO}(g)$ acts transitively on $Q$ (ref. [19], §4, theor. 1, cor. 2). The kernel of inefficiency of this action is

$$J = \{I\} \text{ for } n \text{ odd}, \qquad J = \{I, -I\} \text{ for } n \text{ even}. \tag{3.8}$$

The effective *Möbius group* of transformations of $Q$ is $\mathrm{SO}(g)/J$.

If $V$ is a vector subspace of $W$, then $V^\perp$ is the subspace orthogonal to $V$. Since $g$ is non-degenerate, $V^{\perp\perp} = V$. If $a \in \mathrm{SO}(g)$, then

$$aV = V \text{ is equivalent to } aV^\perp = V^\perp. \tag{3.9}$$

If $V \subset V^\perp$, then all elements of $V$ have vanishing squares; such vectors, and $V$ itself, are said to be *null* (sometimes: isotropic). The subbundle $F$ of $E$ admits an orthogonal subbundle $F^\perp$ relative to the fibre metric $g$ on $E$:

$$F^\perp = \{([w], v) \in Q \times W : g(w,v) = 0\}.$$

As $g(w,w) = 0$, $F \subset F^\perp$.

**Proposition 1.** *There is a natural isomorphism of vector bundles over* $Q$,

$$TQ \cong \mathrm{Hom}(F, F^\perp/F).$$

This is simply a reformulation of (3.4), using the condition that $[(w,v)] \in TQ$ if and only if $g(w,v) = 0$. The scalar product $g$ induces a scalar product in the fibres of the bundle $F^\perp/F \to Q$, but the tangent bundle $TQ$ inherits only a weaker *conformal structure*, which is preserved by the action of the Möbius group.

### 3.3. CONFORMAL COMPACTIFICATION

If $w_0, w_\infty \in W$ are as in (3.6), then

$$V = \operatorname{span}\{w_0, w_\infty\}^\perp \tag{3.10}$$

is an $n$-dimensional vector subspace of $W$ and there is a decomposition of $W$ into a direct sum of vector spaces,

$$W = V \oplus K w_0 \oplus K w_\infty,$$

such that $(K w_\infty)^\perp = V \oplus K w_\infty$. The restriction $h$ of $g$ to $V$ is non-degenerate; if $K = \mathbb{R}$ and the signature of $g$ is $(p+1, q+1)$, then the signature of $h$ is $(p, q)$. The injection

$$i : V \to Q, \quad i(v) = [v + w_0 - g(v,v) w_\infty], \tag{3.11}$$

is smooth and its image is open and dense in $Q$; more precisely, the complement of the image is the set

$$Q_\infty = \{[w] \in Q : g(w, w_\infty) = 0\}. \tag{3.12}$$

If $K = \mathbb{R}$ and $p$ or $q = 0$, then $Q_\infty = [w_\infty]$ is zero-dimensional, the quadric $Q$ is diffeomorphic to the $n$-dimensional sphere $S_n$, which is a one-point conformal compactification of $V = \mathbb{R}^n$ with $h$ the standard scalar product. If $K = \mathbb{C}$ or $K = \mathbb{R}$ and $pq \geq 1$, then $Q_\infty$ is the $(n-1)$-dimensional compactified "light cone at infinity" [1, 20].

If $a \in SO(g)$, $v \in V$ and $i(v) = [w]$, then from the definition of the action of $SO(g)$ on $Q$, one has $ai(v) = [aw]$. The vector $av$ need not be in $V$ and, even if it is, $i(av)$ need not coincide with $ai(v)$.

**Proposition 2.** *The map $i$ is a conformal diffeomorphism of $V$ on its image in $Q$.*

To prove the proposition, evaluate the tangent map $i_* : TV \to TQ$ using $TV \cong V \times V$ and considering a curve $\mathbb{R} \ni t \mapsto i(u + tv) \in Q$, where $u, v \in V$. Computing the tangent vector to the curve at $t = 0$ one obtains

$$i_*(u, v) = [(u + w_0 - g(u,u) w_\infty, v - 2g(u,v) w_\infty)]. \tag{3.13}$$

Since $v$ is orthogonal to the null vector $w_\infty$, one sees that $i$ is conformal.

If $[w] \in i(V)$, then $g(w, w_\infty) \neq 0$. The inverse of (3.11) is the smooth map $i^{-1} : i(V) \to V$ given by

$$i^{-1}([w]) = (\tfrac{1}{2} w - g(w, w_\infty) w_0 - g(w, w_0) w_\infty) / g(w, w_\infty).$$

In particular, $i^{-1}([w_0]) = 0$, whereas $[w_\infty]$ does not belong to $i(V)$: it is a "point at infinity" with respect to $V$. Note that, since $V$ is *conformally flat*, so is $Q$.

### 3.4. THE LINEAR ISOTROPY REPRESENTATION

The restriction $h$ of $g$ to $V$ gives rise to the special orthogonal group $\mathrm{SO}(h)$. The *conformal group* of $h$ is

$$\mathrm{CO}(h) = \{a \in \mathrm{GL}(V) : \exists l \in K' \text{ such that } v \in V \Rightarrow g(av, av) = l^2 g(v,v)\}.$$

There is the exact sequence

$$1 \to J \to K' \times \mathrm{SO}(h) \xrightarrow{f_0} \mathrm{CO}(h) \to 1,$$

where

$$f_0(l,a)v = lav, \quad (l,a,v) \in K' \times \mathrm{SO}(h) \times V,$$

$J$ is as in (3.8) and $K'$ is the multiplicative group of non-zero elements of $K$. The homomorphism $f_0$ makes

$$H = K' \times \mathrm{SO}(h) \times V \tag{3.14}$$

into a semi-direct product of $K' \times \mathrm{SO}(h)$ by $V$: the composition of elements in $H$ is given by

$$(l,a,v)(l',a',v') = (ll', aa', f(l,a,v)v'), \tag{3.15}$$

where

$$f(l,a,v)v' = v + f_0(l,a)v'. \tag{3.16}$$

The map $v' \mapsto f(l,a,v)v'$ defines an action of $H$ in $V$ which is a composition of a rotation $a$, dilatation by $l$ and translation by $v$.

Let

$$H_\infty = \{a \in \mathrm{SO}(g) : a[w_\infty] = [w_\infty]\}$$

be the isotropy subgroup of $\mathrm{SO}(g)$ at $[w_\infty] \in Q$. As $W$ admits a linear frame composed of null vectors, an element $a \in \mathrm{SO}(g)$ is uniquely determined by its action on $N$. Hence $a \in H_\infty$ is equivalent to $aQ_\infty = Q_\infty$ and also to $ai(V) = i(V)$. If $a \in H_\infty$, then there is $\lambda(a) \in K'$ such that

$$aw_\infty = \lambda(a)w_\infty. \tag{3.17}$$

Since $[w_0] \in i(V)$ and $g(aw_0, aw_\infty) = g(w_0, w_\infty)$, there is a vector $\nu(a) \in V$ such that

$$aw_0 = \lambda(a)^{-1}(\nu(a) + w_0 - h(\nu(a),\nu(a))w_\infty). \tag{3.18}$$

If $v \in V$, then $\lambda(a)g(av, w_\infty) = g(av, aw_\infty) = g(v, w_\infty) = 0$ and $g(av, aw_0) = g(v, w_0) = 0$. There thus exists $\mu(a) \in \mathrm{O}(h)$ such that

$$av = \mu(a)v - 2h(\mu(a)v, \nu(a))w_\infty. \tag{3.19}$$

Checking that $\det \mu(a) = \det a$ one establishes $\mu(a) \in \mathrm{SO}(h)$ and proves that the map

$$H_\infty \to H, \quad a \mapsto (\lambda(a), \mu(a), \nu(a))$$

is an isomorphism of groups. The action of $H_\infty$ in $i(V)$ induces the action (3.16) of $H$ in $V$: this is expressed by the relation

$$a \circ i = i \circ f(\lambda(a), \mu(a), \nu(a)), \tag{3.20}$$

easy to verify from (3.11), (3.17)–(3.19) and $a[w] = [aw]$.

Elements of $O(g)$ not in $H_\infty$ induce in $V$ *local* conformal transformations. For example, the map $a : W \to W$ given by

$$aw_0 = w_\infty, \qquad aw_\infty = w_0, \qquad av = v \quad \text{for } v \in V,$$

is orthogonal. Assuming $h(v, v) \neq 0$ one has

$$a[v + w_0 - h(v, v)w_\infty] = [-v/h(v, v) + w_0 - w_\infty/h(v, v)],$$

i.e., $a$ induces the *inversion* $v \mapsto -v/h(v, v)$ defined on $V$ with its light cone removed.

The tangent space $T_\infty Q$ to $Q$ at $[w_\infty]$ is isomorphic to $V$; an isomorphism $j$ is given by

$$j : V \to T_\infty Q, \quad j(v) = [(w_\infty, v)]. \tag{3.21}$$

The tangent action of $H_\infty$ in $T_\infty Q$, i.e. the *linear isotropy representation* of $H_\infty$, is given by (3.5) and, through (3.21), induces an action of $CO(h)$ in $V$. This is expressed by

$$a_* \circ j = j \circ f_0(\lambda(a)^{-1}, \mu(a)). \tag{3.22}$$

### 3.5. COMPLEX QUADRICS

Complex quadrics are outside the scope of this paper and we restrict ourselves to a few remarks on their properties. The *complex quadric* of complex dimension $n$,

$$Q_n = \{[z] \in \mathbb{CP}_{n+1} : (z|z) = 0\}, \tag{3.23}$$

is diffeomorphic to the Grassmannian $\tilde{G}(2, n)$ of oriented two-planes in $\mathbb{R}^{n+2}$. The complex quadric has two "natural" geometric structures: the (holomorphic) conformal geometry induced by the quadratic form $(z|z)$ in $\mathbb{C}^{n+2}$ as described in section 3.3 and the Kähler structure induced by the Hermitean form $(z|\bar{z})$. The latter structure can be thought of as induced by the embedding of $Q_n$ in $\mathbb{CP}_{n+1}$ equipped with the Fubini–Study metric [12]. Since $\tilde{G}(p, n) = SO(n + p)/(SO(n) \times SO(p))$, the quadric $Q_n$, considered as a proper Riemannian space, is a spin manifold if, and only if, $n = 1$ or $n$ is even [11, th. 8].

Complex quadrics do not admit any smooth, complex-bilinear Riemannian metric. This may be seen as follows [21]: such a metric would define a complex-linear isomorphism of the tangent bundle $T = TQ_n$ on the dual bundle $T^*$. The Kähler metric—which exists—defines an isomorphism of $T^*$ with the complex conjugate bundle $\bar{T}$. For the first Chern classes one has $c_1(\bar{T}) = -c_1(T)$ and $c_1(T) \neq 0$ for complex quadrics [12]. Therefore, $T$ and $T^*$ cannot be isomorphic.

## 4. The real quadrics

From now on we assume $K = \mathbb{R}$ and consider only the real quadrics. Let $X = \mathbb{R}^{p+1}$, $Y = \mathbb{R}^{q+1}$ and $W = X \times Y$ be given a scalar product $g$ of signature $(p+1, q+1)$ such that

$$g(w,w) = (x|x) - (y|y), \quad w = (x,y) \in X \times Y.$$

The *real quadric*

$$Q_{p,q} = \{ [(x,y)] : (x,y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}, (x|x) = (y|y) \neq 0\} \tag{4.1}$$

is a compact, $(p+q)$-dimensional submanifold of the real projective space $\mathbb{RP}_{n+1} = \mathrm{S}_{n+1}/\mathbb{Z}_2$, where $n = p+q$. The embedding

$$Q_{p,q} \to Q_{p+q}, \quad [(x,y)] \mapsto [(x, \sqrt{-1}y)]$$

defines $Q_{p,q}$ as a "real form" of $Q_{p+q}$, cf. (3.23).

Since every line $[(x,y)] \in Q_{p,q}$ contains exactly two opposite vectors, say $(x,y)$ and $(-x,-y)$, such that $(x|x) = (y|y) = 1$, the quadric $Q_{p,q}$ may be identified with the quotient $(\mathrm{S}_p \times \mathrm{S}_q)/\mathbb{Z}_2$ and the embedding

$$(\mathrm{S}_p \times \mathrm{S}_q)/\mathbb{Z}_2 = Q_{p,q} \to \mathbb{RP}_{p+q+1} = \mathrm{S}_{p+q+1}/\mathbb{Z}_2 \tag{4.2}$$

comes from the tautological map

$$\mathrm{S}_p \times \mathrm{S}_q \to \mathrm{S}_{p+q+1} \tag{4.3}$$

sending $(x,y) \in \mathrm{S}_p \times \mathrm{S}_q$ to $(x/\sqrt{2}, y/\sqrt{2}) \in \mathrm{S}_{p+q+1}$. Clearly, $Q_{p,0} \cong Q_{0,p} \cong \mathrm{S}_p$. From now on we consider only *proper* quadrics, i.e., such that $pq \geq 1$.

The connected component $\mathrm{SO}^0(p+1, q+1)$ of the group of orthogonal transformations of $W$ acts transitively and conformally on $Q_{p,q}$. This group can be restricted—without loss of transitivity—to its maximal compact subgroup

$$G = \mathrm{SO}(p+1) \times \mathrm{SO}(q+1). \tag{4.4}$$

The structure of the tangent bundle $T_{p,q} = TQ_{p,q}$ can be obtained from proposition 1 or directly from that of $T(\mathrm{S}_p \times \mathrm{S}_q)$. Namely, we have

**Proposition 3.** *The tangent bundle $T_{p,q}$ of the proper quadric $Q_{p,q}$ decomposes into the direct sum of two vector bundles $T'_{p,q}$ and $T''_{p,q}$ with fibre dimensions $p$ and $q$, respectively. Each of these subbundles is orientable iff its fibre dimension is odd. The bundle $T_{p,q}$ is orientable iff $p+q$ is even.*

*Proof.* The tangent bundle of the $p$-sphere $\mathrm{S}_p$ is

$$T\mathrm{S}_p = \{(x,u) \in \mathrm{S}_p \times \mathbb{R}^{p+1} : (x|u) = 0\}.$$

Consider the action of the group $\mathbb{Z}_2 = \{1, -1\}$ in $T\mathrm{S}_p \times T\mathrm{S}_q$ given by

$$((x,u),(y,v))(-1) = ((-x,-u),(-y,-v));$$

then

$$T_{p,q} = (T\mathrm{S}_p \times T\mathrm{S}_q)/\mathbb{Z}_2.$$

The formula

$$[(x,u),(y,v)] = [(x,u),(y,0)] + [(x,0),(y,v)]$$

defines the subbundles $T'_{p,q}$ and $T''_{p,q}$ and the decomposition

$$T_{p,q} = T'_{p,q} \oplus T''_{p,q}. \tag{4.5}$$

Consider now the canonical projections

$$\mathrm{pr}_1 : \mathrm{S}_p \times \mathrm{S}_q \to \mathrm{S}_p, \quad \mathrm{pr}_2 : \mathrm{S}_p \times \mathrm{S}_q \to \mathrm{S}_q,$$

and

$$\pi : \mathrm{S}_p \times \mathrm{S}_q \to (\mathrm{S}_p \times \mathrm{S}_q)/\mathbb{Z}_2, \quad \pi(x,y) = [(x,y)]. \tag{4.6}$$

Let $\alpha_p$ be the antipodal map, $\alpha_p : \mathrm{S}_p \to \mathrm{S}_p$, $\alpha_p(x) = -x$. If $\omega_p$ is the standard volume $p$-form on $\mathrm{S}_p$, then $\alpha_p^*\omega_p = (-1)^{p+1}\omega_p$ and, whenever $p$ is odd, the form $\omega_p$ defines a nowhere vanishing section of the bundle $\wedge^p T'^*_{p,q}$, thus proving the orientability of $T'_{p,q}$. Moreover,

$$\omega_{p,q} = \mathrm{pr}_1^*\omega_p \wedge \mathrm{pr}_2^*\omega_q$$

is a $(p+q)$-form on $\mathrm{S}_p \times \mathrm{S}_q$ such that

$$\alpha_{p,q}^*\omega_{p,q} = (-1)^{p+q}\omega_{p,q},$$

where

$$\alpha_{p,q} : \mathrm{S}_p \times \mathrm{S}_q \to \mathrm{S}_p \times \mathrm{S}_q, \quad \alpha_{p,q}(x,y) = (-x,-y). \tag{4.7}$$

Therefore, if $p+q$ is even, then the form $\omega_{p,q}$ descends to a volume form on the quadric, thus proving its orientability. Conversely, if the quadric is orientable, then, since $G$ is compact and connected, there is a volume form $\omega$ on the quadric, invariant with respect to $G$. The form $\pi^*\omega$ on $\mathrm{S}_p \times \mathrm{S}_q$ is then also invariant with respect to $G$ and $\alpha_{p,q}^*\pi^*\omega = \pi^*\omega$; therefore, $\pi^*\omega$ is proportional to $\omega_{p,q}$ and $p+q$ is even. A similar argument shows that $T'_{p,q}$ ($T''_{p,q}$) is orientable only if $p$ ($q$) is odd. □

The group $G$ acts as a transitive group of isometries of two distinguished Riemannian metrics $g^+_{p,q}$ and $g^-_{p,q}$ on $Q_{p,q}$; they can be described as follows. Let $g_n$ denote the standard proper Riemannian metric on $\mathrm{S}_n$; the metrics $g^\pm_{p,q}$ are characterized by

$$\pi^* g^\pm_{p,q} = \mathrm{pr}_1^* g_p \pm \mathrm{pr}_2^* g_q. \tag{4.8}$$

The metric $g^+_{p,q}$ coincides, up to a numerical factor, with the restriction to $Q_{p,q}$ of the Fubini–Study metric on $\mathbb{CP}_{p+q+1}$. The conformal class of the metric $g^-_{p,q}$ defines the conformal geometry on $Q_{p,q}$ described in section 3.

From now on, throughout the paper, we assume

$$p + q = 2m, \quad m = 1, 2, \ldots. \tag{4.9}$$

By virtue of this assumption, the integers $p$ and $q$ are simultaneously even or odd; we refer to these two *cases* as *even* and *odd*, respectively.

**Proposition 4.** *In the even case, the action of the group* $G = \mathrm{SO}(p+1) \times \mathrm{SO}(q+1)$ *on* $Q_{p,q}$ *is effective; in the odd case, the kernel of inefficiency* $J$ *is* $\mathbb{Z}_2$, *generated by* $(-I_{p+1}, -I_{q+1}) \in G$, *where* $I_n$ *denotes the identity in* $\mathbb{R}^n$.

This is a straightforward consequence of the fact that $-I_n$ belongs to $\mathrm{SO}(n)$ iff $n$ is even.

To determine the isotropy group of a point in $Q_{p,q}$ under the action of $G$ it is convenient to consider the monomorphism

$$h_p : \mathrm{O}(p) \to \mathrm{SO}(p+1) \tag{4.10}$$

characterized by

$$\begin{aligned} h_p(A)e_i &= Ae_i, \quad i = 1, \ldots, p, \\ h_p(A)e_{p+1} &= (\det A)e_{p+1}. \end{aligned} \tag{4.11}$$

An element of $\mathrm{SO}(p+1)$ is in the image of $h_p$ if, and only if, it commutes with the reflection $E_{p+1}$ of $\mathbb{R}^{p+1}$ in the hyperplane orthogonal to $e_{p+1}$. The reflection of $\mathbb{R}^{q+1}$ in the hyperplane orthogonal to $f_{q+1}$ is denoted by $F_{q+1}$.

**Proposition 5.** *The subgroup* $H$ *of* $G$ *leaving invariant the point* $[(e_{p+1}, f_{q+1})] \in Q_{p,q}$ *is isomorphic to the group*

$$\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q)) = \{(A, B) \in \mathrm{O}(p) \times \mathrm{O}(q) : \det A = \det B\}.$$

*In the odd case, the isotropy subgroup of the effective group* $G/\mathbb{Z}_2$ *is isomorphic to* $\mathrm{SO}(p) \times \mathrm{SO}(q)$.

Indeed, if $(A', B') \in H$, then either $A'e_{p+1} = e_{p+1}$ and $B'f_{q+1} = f_{q+1}$ or $A'e_{p+1} = -e_{p+1}$ and $B'f_{q+1} = -f_{q+1}$. Therefore, there is a pair $(A, B) \in \mathrm{O}(p) \times \mathrm{O}(q)$ such that $A' = h_p(A)$, $B' = h_q(B)$ and $\det A = \det B$.

In the odd case, $\det(-A) = -\det A$ for $A \in \mathrm{O}(p)$ or $\mathrm{O}(q)$ and there is the commutative diagram of group homomorphisms

$$\begin{array}{ccc} \mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q)) & \overset{h_p \times h_q}{\longrightarrow} & \mathrm{SO}(p+1) \times \mathrm{SO}(q+1) \\ \downarrow & & \downarrow \\ \mathrm{SO}(p) \times \mathrm{SO}(q) & \longrightarrow & (\mathrm{SO}(p+1) \times \mathrm{SO}(q+1))/\mathbb{Z}_2 \end{array}$$

where the first vertical arrow is $(A, B) \mapsto (\varepsilon A, \varepsilon B)$, $\varepsilon = \det A = \det B$, and the lower horizontal arrow is injective.

It is also worth noting that

$$\mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(q)) = \mathrm{SO}(p+q)\cap\mathrm{SO}(p,q), \tag{4.12}$$

$$\mathrm{SO}(p)\times\mathrm{SO}(q) = \mathrm{SO}(p+q)\cap\mathrm{SO}^0(p,q). \tag{4.13}$$

The map

$$\sigma : G\to G,\quad \sigma(A,B) = (E_{p+1}AE_{p+1}, F_{q+1}BF_{q+1})$$

is an involutive automorphism of $G$. The subgroup $G^\sigma$ of $G$ consisting of all elements left invariant by $\sigma$ is isomorphic to $\mathrm{O}(p)\times\mathrm{O}(q)$ and its connected component of the unit element is $G_0^\sigma = \mathrm{SO}(p)\times\mathrm{SO}(q)$. This proves (see ref. [12], ch. XI, §2):

**Proposition 6.** *The triple $(G,H,\sigma)$ is a symmetric space,*

$$G_0^\sigma\subset H\subset G^\sigma,$$

*for either of the two metrics $g^+_{p,q}$ and $g^-_{p,q}$ on $Q_{p,q}$.*

The tangent space to $Q_{p,q}$ at $[(e_{p+1}, f_{q+1})]$ is isomorphic to $\mathbb{R}^p\times\mathbb{R}^q$. Depending on whether the quadric is given the metric $g^+_{p,q}$ or $g^-_{p,q}$, this vector space inherits a bilinear form of signature $(p+q,0)$ or $(p,q)$, respectively. The action of the effective isotropy group on tangent vectors induces an injection of this group into $\mathrm{SO}(p+q)$ or $\mathrm{SO}(p,q)$, respectively. A simple computation, based on proposition 3, or on the canonical decomposition of the symmetric Lie algebra associated with the symmetric space $(G,H,\sigma)$, leads to

**Proposition 7.** *The tangent action of the isotropy group $H$ of a point on the quadric $Q_{p,q}$ induces the automorphism*

$$\mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(q))\to\mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(q)) \tag{4.14}$$

*given by*

$$(A,B)\mapsto(\varepsilon A,\varepsilon B), \tag{4.15}$$

*where $\varepsilon = \det A = \det B$.*

Since $\det\varepsilon A = \det\varepsilon B = \varepsilon^{p+1}$, in the odd case—unlike in the even one—the effective isotropy group injects into the connected component of the identity of the pseudo-orthogonal group. This is so because only in the odd case is the quadric not only orientable, but also space- and time-orientable: the subbundles $T'_{p,q}$ and $T''_{p,q}$ are both orientable (proposition 3).

Recall that in the "Lorentzian case", i.e. when $p$ and $q$ are both odd and $p$ or $q = 1$, the quadric is a Cartesian product of a circle by a sphere. Indeed, let $p =$

1 and $q = 2m - 1$ and represent an odd-dimensional sphere as a hypersurface in $\mathbb{C}^m$,

$$S_{2m-1} = \{z \in \mathbb{C}^{2m} : (z|\bar{z}) = 1\}.$$

If $z \in S_{2m-1}$ and $z_0 \in S_1$, i.e. $z_0 \in \mathbb{C}$ and $|z_0| = 1$, then $z_0 z \in S_{2m-1}$ and the map

$$(S_1 \times S_{2m-1})/\mathbb{Z}_2 \to S_1 \times S_{2m-1}, \quad [(z_0, z)] \mapsto (z_0^2, z_0 z) \tag{4.16}$$

is a diffeomorphism.

## 5. Spin structures on homogeneous spaces

### 5.1. DEFINITIONS

We recall now the definition of a *spin structure* on a connected, oriented $n$-dimensional proper Riemannian manifold $M$. Let $\pi : P \to M$ be the principal $SO(n)$-bundle of orthonormal frames of the given orientation on $M$. A *spin structure* on $M$ is a prolongation (section 2.1) $\sigma : \hat{P} \to P$ of $P$ to $\mathrm{Spin}(n)$, corresponding to $\rho : \mathrm{Spin}(n) \to SO(n)$. The map $\sigma$ makes $\hat{P}$ into a double cover of $P$ and $\pi \circ \sigma : \hat{P} \to M$ is a principal $\mathrm{Spin}(n)$-bundle.

A Riemannian manifold with a given spin structure is said to be a *spin manifold.*

It is clear that the definition of a spin structure extends, *mutatis mutandis*, to pseudo-Riemannian manifolds. If such a manifold is not only orientable, but also space- and time-orientable, then its bundle of frames can be restricted to $SO^0(p,q)$ and, if $M$ has a spin structure, then the bundle of "spin frames" $\hat{P}$ can be restricted to $\mathrm{Spin}^0(p,q)$. To distinguish between different signatures and orientabilities, we shall use expressions such as $\mathrm{Spin}(n)$-structure, $\mathrm{Spin}(p,q)$-structure and $\mathrm{Spin}^0(p,q)$-structure. There are similar structures corresponding to non-orientable and space- or time-orientable manifolds (see ref. [15], but note that Satz 2.2 on p. 71 should be replaced by the condition for the existence of spin structures on pseudo-Riemannian manifolds given by Karoubi [16], cf. section 7).

### 5.2. BUNDLE OF ORTHONORMAL FRAMES

The bundle of orthonormal frames of a homogeneous Riemannian manifold can be described as follows.

**Proposition 8.** *Let $M$ be an $n$-dimensional, oriented proper Riemannian manifold with a transitive Lie group $G$ of orientation-preserving isometries. Let $H$ be the isotropy subgroup of $G$ at the point $o \in M$ and $\tau : H \to SO(n)$ be the linear isotropy representation defined by the tangent action of $H$ on vectors. The bundle*

*P of orthonormal frames of the given orientation on M is isomorphic to the bundle associated by $\tau$ with the principal H-bundle $G \to G/H = M$. The action of G on M lifts to an action to the left on P, commuting with the action of* $\mathrm{SO}(n)$.

*Proof.* The representation $\tau$ can be described as follows. Let $\xi_o : \mathbb{R}^n \to T_oM$ be an orthonormal frame at $o$, i.e., an isometry from $\mathbb{R}^n$, with its standard scalar product, to $T_oM$ endowed with the bilinear form obtained by restriction of the metric tensor on $M$. If $a_* : TM \to TM$ is the action of $a \in G$ on tangent vectors, then, for $c \in H$, we have $\tau(c) = \xi_o^{-1} \circ c_* \circ \xi_o$. The bundle $G \times_\tau \mathrm{SO}(n)$ associated with $G \to G/H$ by $\tau$ consists of equivalence classes $[(a,b)]$, where $(a,b) \in G \times \mathrm{SO}(n)$ and $[(a,b)] = [(a',b')]$ whenever there exists $c \in H$ such that $a' = ac$ and $b = \tau(c)b'$. The map

$$G \times_\tau \mathrm{SO}(n) \to P, \quad [(a,b)] \mapsto a_* \circ \xi_o \circ b \tag{5.1}$$

is well defined, bijective and equivariant with respect to the action of $\mathrm{SO}(n)$ in both bundles: it defines the isomorphism whose existence is asserted in the proposition. The lift of the action of $G$ to $P$ is given by $a[(b,c)] = [(ab,c)]$, where $a, b \in G$ and $c \in \mathrm{SO}(n)$. □

It is clear that proposition 8 extends, in an obvious manner, to pseudo-Riemannian manifolds. In the sequel, we formulate several propositions about spin structures associated with homogeneous, proper Riemannian manifolds. The assumption that the metric is positive-definite is made to alleviate the exposition. It will be clear that our considerations can be extended to homogeneous, pseudo-Riemannian manifolds. In fact, such extensions will have to be used in the applications of the general theorems to the real quadrics $Q_{p,q}$ with the metric tensor $g_{p,q}^-$.

### 5.3. LIFTS TO SPIN

Consider the homomorphism

$$\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n),$$

defined in section 2.2, a Lie group $H$ and a homomorphism

$$\tau : H \to \mathrm{SO}(n). \tag{5.2}$$

We say that the homomorphism

$$\hat{\tau} : H \to \mathrm{Spin}(n) \tag{5.3}$$

is a *lift* of $\tau$ to $\mathrm{Spin}(n)$ if $\rho \circ \hat{\tau} = \tau$.

**Lemma 1.** *Any two lifts of a homomorphism (5.2) coincide when restricted to the connected component $H_0$ of the unit element of the Lie group H. If $\hat{\tau}$ and $\hat{\tau}'$ are* *two such lifts, then the map $H \to Spin(n)$ given by $a \mapsto \hat{\tau}'(a)\hat{\tau}(a^{-1})$ takes values in $\mathbb{Z}_2 \subset Spin(n)$ and factors through the canonical map $H \to H/H_0$ to yield a homomorphism $\kappa : H/H_0 \to \mathbb{Z}_2$. Conversely, given such a homomorphism $\kappa$ and a lift $\hat{\tau}$ of $\tau$, the map $\hat{\tau}' : H \to Spin(n)$, given by $\hat{\tau}'(a) = \kappa(aH_0)\hat{\tau}(a)$, is another lift of $\tau$.*

To prove the lemma, it is enough to note that, given two lifts $\hat{\tau}$ and $\hat{\tau}'$ of $\tau$, the map $a \mapsto \hat{\tau}'(a)\hat{\tau}(a^{-1})$ from $H$ to $\mathrm{Spin}(n)$ is a lift of the map $a \mapsto I_n$ from $H$ to $\mathrm{SO}(n)$; therefore, the former map is $\mathbb{Z}_2$-valued and constant over every connected component of $H$.

### 5.4. THE UNIVERSAL COVERING GROUP

The universal covering group of a connected Lie group is simply connected and the following lemma holds [22]:

**Lemma 2.** *The universal covering group $\tilde{G}$ of a connected Lie group $G$ is a principal $\pi_1(G)$-bundle over $G$; the Abelian discrete group $\pi_1(G)$ can be identified with a subgroup of the centre of $\tilde{G}$ and there is an exact sequence*

$$1 \to \pi_1(G) \to \tilde{G} \xrightarrow{\iota} G \to 1. \tag{5.4}$$

*Every double cover $\hat{G}$ of $G$ can be represented as $\tilde{G} \times_\theta \mathbb{Z}_2$, where $\theta : \pi_1(G) \to \mathbb{Z}_2$ is a homomorphism. In particular, if $\theta$ is surjective, then $\hat{G} = \tilde{G}/\ker\theta$ is connected; otherwise, $\hat{G} = G \times \mathbb{Z}_2$.*

### 5.5. CONSTRUCTION OF SPIN STRUCTURES

Consider two spin manifolds $M'$ and $M''$ of dimension $p$ and $q$, respectively. The metric tensors $g'$ on $M'$ and $g''$ on $M''$ define two metric tensors $g^+$ and $g^-$ on the product manifold $M = M' \times M''$,

$$g^\pm = \mathrm{pr}_1^* g' \pm \mathrm{pr}_2^* g''.$$

Let $\sigma' : \hat{P}' \to P'$ be the spin structure on $M'$, where $P'$ is the $\mathrm{SO}(p)$-bundle of orthonormal frames on $M'$ and let a similar notation apply to $M''$. The principal $\mathrm{SO}(p) \times \mathrm{SO}(q)$-bundle $P' \times P'' \to M' \times M''$ can be considered as a restriction of the $\mathrm{SO}(p+q)$-bundle ($\mathrm{SO}^0(p,q)$-bundle) of frames on $M$, orthonormal with respect to $g^+$ ($g^-$). The manifold $\hat{P}' \times \hat{P}''$ is a fourfold cover of $P' \times P''$ and the bundles

$$(\hat{P}' \times \hat{P}'') \times_{\varphi_+} \mathrm{Spin}(p+q), \qquad (\hat{P}' \times \hat{P}'') \times_{\varphi_-} \mathrm{Spin}^0(p,q),$$

where the homomorphisms $\varphi_+$ and $\varphi_-$ are as in (2.9) and (2.10), define spin structures on $M$, associated with $g^+$ and $g^-$, respectively. This observation can be used to construct spin structures on the quadrics $Q_{1,2m-1} \cong S_1 \times S_{2m-1}$. In the general case, we can appeal to

**Theorem 1.** *Let $M$ be an $n$-dimensional, oriented, connected Riemannian manifold with a transitive Lie group $G$ of orientation-preserving isometries. Let $H$ be the isotropy group of a point $o$ of $M$ and $\tau : H \to \mathrm{SO}(n)$ be the linear isotropy representation. Then*

*(i) if $\tau$ lifts to $\hat{\tau} : H \to Spin(n)$, then there is a spin structure on $M$ such that $\hat{P} = G \times_{\hat{\tau}} Spin(n)$;*

*(ii) if $\hat{\tau}$ and $\hat{\tau}'$ are two lifts of $\tau$ and the spin structures defined by $\hat{P}$ and $\hat{P}' = G \times_{\hat{\tau}'} Spin(n)$ are isomorphic, then $\hat{\tau} = \hat{\tau}'$;*

*(iii) if the group $G$ is simply connected and $M$ has a spin structure, then $\tau$ lifts to $Spin(n)$.*

*Proof.*

(i) The homomorphism $\hat{\tau} : H \to \mathrm{Spin}(n)$ defines the principal $\mathrm{Spin}(n)$-bundle $\hat{P} = G \times_{\hat{\tau}} \mathrm{Spin}(n) \to M$ as a bundle associated with the $H$-bundle $G \to G/H = M$. The map $\sigma : \hat{P} \to P = G \times_{\tau} \mathrm{SO}(n)$ is well defined by $\sigma[(a,b)] = [(a,\rho(b))]$, where $(a,b) \in G \times \mathrm{Spin}(n)$. The standard definition of the action of $\mathrm{Spin}(n)$ in $\hat{P}$ gives $\sigma([(a,b)]c) = \sigma[(a,bc)] = [(a,\rho(b))]\rho(c)$, where $a \in G$ and $b,c \in \mathrm{Spin}(n)$.

(ii) Let $\lambda : \hat{P} \to \hat{P}'$ be an isomorphism of the spin structure $\sigma : \hat{P} \to P$ onto $\sigma' : \hat{P}' \to P$, i.e., a diffeomorphism such that $\sigma' \circ \lambda = \sigma$ and $\lambda(\eta b) = \lambda(\eta)b$ for every $\eta \in \hat{P}$ and $b \in \mathrm{Spin}(n)$. The first of these conditions is equivalent to the existence of a map $\varepsilon : \hat{P} \to \mathbb{Z}_2 \subset \mathrm{Spin}(n)$ such that $\lambda[(a,b)] = [(a,\varepsilon[(a,b)]b)]$. The second implies that $\varepsilon$ is constant on the fibres of $\hat{P} \to M$ and, therefore, defines a continuous function $M \to \mathbb{Z}_2$ which is constant by virtue of the connectedness of $M$. Thus $\lambda[(a,b)] = \lambda[(a,\epsilon b)]$, where $\epsilon = 1$ or $-1$. On the other hand, the chain of equalities

$$\begin{aligned}[(ac,\epsilon b)] &= \lambda[(ac,b)] = \lambda[(a,\hat{\tau}(c)b)] \\ &= [(a,\epsilon\hat{\tau}(c)b)] = [(ac,\epsilon\hat{\tau}'(c^{-1})\hat{\tau}(c)b)]\end{aligned}$$

gives $\hat{\tau} = \hat{\tau}'$.

(iii) Let $\sigma : \hat{P} \to P = G \times_{\tau} \mathrm{SO}(n) \xrightarrow{\pi} M$ be a spin structure on $M$. Since $G$ is simply connected, by the homotopy lifting theorem for covering spaces [22], the action of $G$ on $P$ lifts to an action of $G$ on $\hat{P}$, $(a,\eta) \mapsto a\eta$, commuting with that of $\mathrm{Spin}(n)$ and such that $\sigma(a\eta) = a\sigma(\eta)$, where $a \in G$ and $\eta \in \hat{P}$. Let $\xi_o \in P$ be the orthonormal frame at $o \in M$, occurring in the definition of the representation $\tau$, cf. the proof of proposition 8. Let $\eta_o$ be one of the two elements of $\hat{P}$ such that $\sigma(\eta_o) = \xi_o$. Since the action of $\mathrm{Spin}(n)$ is transitive on the fibres of $\pi \circ \sigma : \hat{P} \to M$ and, for every $a \in H$, the elements $\eta_o$ and $a\eta_o$ are in the same fibre, there exists an element $\hat{\tau}(a)$ of $\mathrm{Spin}(n)$ such that

$a\eta_o = \eta_o\hat{\tau}(a)$. By the commutativity of the actions of $G$ and $\mathrm{Spin}(n)$ in $\hat{P}$, this element does not change when $\eta_o$ is replaced by $\eta_o \cdot (-1)$ and $\hat{\tau} : H \to \mathrm{Spin}(n)$ is a homomorphism of groups. The equivariance condition $\sigma(\eta c) = \sigma(\eta)\rho(c)$ where $\eta \in \hat{P}$ and $c \in \mathrm{Spin}(n)$, shows that $\rho \circ \hat{\tau} = \tau$. □

Any transitive Lie group of transformations of a manifold can be replaced by a Lie group acting transitively and effectively: it suffices to quotient the group by its kernel of inefficiency.

**Theorem 2.** *Let $M, G$ and $H$ be as in theorem 1. Assume furthermore that $G$ is connected and acts effectively; let $\tilde{G}$ be the universal covering group of $G$ and $\tilde{H}$ be the subgroup of $\tilde{G}$ covering $H$. Then*

*(i) the linear isotropy representation $\tau : H \to \mathrm{SO}(n)$ is injective and the principal bundle $G \to M$ is a restriction of the principal $\mathrm{SO}(n)$-bundle $P \to M$ to $H$;*

*(ii) if $\tilde{\tau} : \tilde{H} \to Spin(n)$ is a lift of the linear isotropy representation of $\tilde{H}$, then the spin structure $\tilde{G} \times_{\tilde{\tau}} Spin(n)$ admits a restriction $\hat{G} = \tilde{G} \times_\theta \mathbb{Z}_2$ to the subgroup $\hat{H} = \tilde{H} \times_\theta \mathbb{Z}_2$ of $Spin(n)$, where $\theta : \pi_1(G) \to \mathbb{Z}_2$ is the homomorphism obtained by restricting $\tilde{\tau}$ to $\pi_1(G) \subset \tilde{H}$;*

*(iii) conversely, if there is a spin structure $\hat{P}$ on $M$, and $\hat{G}$ is the restriction of $\hat{P}$ to the subgroup $\hat{H}$ of $Spin(n)$ covering the subgroup $H$ of $\mathrm{SO}(n)$, then there is a homomorphism $\theta : \pi_1(G) \to \mathbb{Z}_2$ such that $\hat{G} = \tilde{G} \times_\theta \mathbb{Z}_2$, $\hat{H} = \tilde{H} \times_\theta \mathbb{Z}_2$ and the linear isotropy representation of $\tilde{H}$ lifts to $\tilde{\tau} : \tilde{H} \to Spin(n)$ obtained by composing the homomorphisms $\tilde{H} \to \hat{H} \to Spin(n)$. The bundle $\hat{P}$ is isomorphic to $\tilde{G} \times_{\tilde{\tau}} Spin(n)$.*

*Proof.*

(i) If $a \in H$ and $a_*|_{T_oM} = \mathrm{id}$, then $a : M \to M$, being an isometry, preserves pointwise every geodesic through $o$; therefore $a$ is ineffective in a neighbourhood of $o$ and, since $M$ is connected and $G$ acts effectively, $a$ is the unit element of $G$; this shows that $\tau$ is injective. The map $G \to P = G \times_\tau \mathrm{SO}(n)$ given by $a \mapsto [(a, I_n)]$ is then injective and makes $G$ into a restriction of $P$ to $H$.

(ii) By restriction, the sequence (5.4) gives the exact sequence

$$1 \to \pi_1(G) \to \tilde{H} \xrightarrow{t} H \to 1.$$

Let $\tau$ be the (injective) linear isotropy representation of $H$; the corresponding representation of $\tilde{H}$ is $\tau \circ t$ and the projection

$$\sigma : \hat{P} = \tilde{G} \times_{\tilde{\tau}} \mathrm{Spin}(n) \to G \times_\tau \mathrm{SO}(n) = P$$

is given by

$$\sigma[(a,b)] = [(t(a), \rho(b))],$$

where $(a,b) \in \tilde{G} \times \mathrm{Spin}(n)$ and $t$ is as in (5.4). Since $\rho \circ \tilde{\tau} = \tau \circ t$, the restriction of $\tilde{\tau}$ to $\pi_1(G) \subset \tilde{H}$ takes values in $\mathbb{Z}_2 \subset \mathrm{Spin}(n)$ and defines a homomorphism

$$\theta : \pi_1(G) \to \mathbb{Z}_2. \tag{5.5}$$

Let $\hat{G}$ and $\hat{H}$ be the groups $\tilde{G} \times_\theta \mathbb{Z}_2$ and $\tilde{H} \times_\theta \mathbb{Z}_2$, respectively. The map

$$\hat{\tau} : \hat{H} \to \mathrm{Spin}(n), \quad \hat{\tau}[(a,\varepsilon)] = \varepsilon\tilde{\tau}(a),$$

where $(a,\varepsilon) \in \tilde{H} \times \mathbb{Z}_2$, is a well-defined homomorphism which is injective,

$$\hat{\tau}[(a,\varepsilon)] = 1 \Rightarrow \varepsilon\tilde{\tau}(a) \Rightarrow a \in \pi_1(G), \qquad \varepsilon = \theta(a^{-1}).$$

The obvious map

$$\hat{G} = \tilde{G} \times_\theta \mathbb{Z}_2 \to \tilde{G} \times_{\tilde{\tau}} \mathrm{Spin}(n) = \hat{P}$$

is also injective and makes $\hat{G}$ into a restriction of $\hat{P}$ to $\hat{H}$.

(iii) If $\hat{P}$ is a spin structure on $M$ and $\hat{G}$ is its restriction to $\hat{H}$, then $\hat{G}$ is a double cover of $G$. According to lemma 2, there exists a homomorphism (5.5) such that $\hat{G} = \tilde{G} \times_\theta \mathbb{Z}_2$. Let $\hat{\tau} : \hat{H} \to \mathrm{Spin}(n)$ be the injection defining $\hat{H}$ as a subgroup of $\mathrm{Spin}(n)$. The linear isotropy representation of $\tilde{H}$ is the composition of maps $\tilde{H} \to H \to \mathrm{SO}(n)$ and it lifts to $\tilde{\tau}$, as announced in the theorem. The bundle $\hat{P}$ is obtained from $\hat{G}$ by extending the structure group of the latter bundle to $\mathrm{Spin}(n)$, $\hat{P} = \hat{H} \times_{\hat{\tau}} \mathrm{Spin}(n)$. Moreover, an easy check shows that the obvious map $\hat{P} \to \tilde{G} \times_{\tilde{\tau}} \mathrm{Spin}(n)$ is an isomorphism of spin structures. □

**Remark.** The relations between the various maps and spaces occurring in theorem 2 are summarized in the commutative diagram

$$\begin{array}{ccccccc}
 & & \tilde{H} & \to & \tilde{G} & & \\
 & {\scriptstyle\tilde{\tau}}\swarrow & \downarrow & & \downarrow{\scriptstyle\delta} & & \\
\mathrm{Spin}(n) & \xleftarrow{\hat{\tau}} & \hat{H} & \to & \hat{G} & \to & \hat{P} = \tilde{G} \times_{\tilde{\tau}} \mathrm{Spin}(n) \\
{\scriptstyle\rho}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{SO}(n) & \xleftarrow{\tau} & H & \to & G & \to & P = G \times_\tau \mathrm{SO}(n)
\end{array}$$

where all horizontal arrows are injective. The homomorphism $\delta : \tilde{G} \to \hat{G} = \tilde{G} \times_\theta \mathbb{Z}_2$, $\delta(a) = [(a,1)]$, has the same kernel as $\theta$ and is surjective whenever $\theta$ is. If $\ker\theta = \pi_1(G)$, then there is a "trivial spin structure", $\hat{G} = G \times \mathbb{Z}_2$.

## 6. Spin structures on even-dimensional real quadrics

Theorem 1 will now be used to determine all spin structures on even-dimensional quadrics. Since the groups $\mathrm{Spin}(n)$ are simply connected for $n \geq 3$, but $\mathrm{Spin}(2)$ is not, we treat separately the quadrics $Q_{1,2m-1}$. We use the notation of theorem 1

(i) Let $p + q = 2m$ and $p, q \geq 2$; the group

$$G = \mathrm{Spin}(p+1) \times \mathrm{Spin}(q+1) \tag{6.1}$$

is simply connected and acts as a transitive group of isometries on $Q_{p,q}$ endowed with either of the two metrics $g^+_{p,q}$ and $g^-_{p,q}$, defined in section 4. In view of the homomorphisms (2.9) and (2.10), one can treat simultaneously both signatures provided that one assumes the generators $(f_1, \ldots, f_{q+1})$ of the Clifford algebra associated with $R^{q+1}$ to satisfy

$$f_\mu f_\nu + f_\nu f_\mu = \pm 2\delta_{\mu\nu}, \tag{6.2}$$

where $\mu, \nu = 1, \ldots, q+1$ and the signs $+$ and $-$ correspond to the signatures $(p+q, 0)$ and $(p, q)$, respectively.

The group $G$ acts in $Q_{p,q}$ according to

$$(a, b)[(x, y)] = [(axa^{-1}, byb^{-1})],$$

where $(a, b) \in G$, $(x, y) \in \mathrm{S}_p \times \mathrm{S}_q$ and Clifford multiplication is understood on the right. The isotropy subgroup $H$ preserving the point $[(e_{p+1}, f_{q+1})]$ has

$$H_0 = \mathrm{Spin}(p) \times \mathrm{Spin}(q)$$

as its connected component of the unit element and is generated by $H_0$ and the element $(e_1 e_{p+1}, f_1 f_{q+1})$. The linear isotropy representation $\tau : H \to \mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$ follows easily from proposition 7. It is given by

$$\tau(a, b) = (\rho(a), \rho(b)) \quad \text{for } (a, b) \in H_0, \tag{6.3}$$

$$\tau(e_1 e_{p+1}, f_1 f_{q+1})) = (-E_1, -F_1), \tag{6.4}$$

where $E_1$ and $F_1$ are the reflections of $\mathbb{R}^p$ and $\mathbb{R}^q$ in hyperplanes orthogonal to $e_1$ and $f_1$, respectively.

The restriction of $\tau$ to $H_0$ lifts to $\hat{\tau} : H_0 \to \mathrm{Spin}(p+q)$ or $\mathrm{Spin}(p, q)$,

$$\hat{\tau}_0(a, b) = ab \quad \text{for } (a, b) \in H_0. \tag{6.5}$$

The element $(-E_1, -F_1) \in \mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$ is covered by the elements $\pm e_1 f_1 \mathrm{vol}$, where $\mathrm{vol} = \mathrm{vol}_{p+q}$ or $\mathrm{vol}_{p,q}$ depending on whether the signature is $(p+q, 0)$ or $(p, q)$, respectively, cf. (2.11) and (2.13). The square of $(e_1 e_{p+1}, f_1 f_{q+1})$ is $(-1, -1) \in H_0$ and is sent to 1 by (6.5). Therefore, a necessary and sufficient condition for the existence of a lift of the linear isotropy representation to Spin is

$$(e_1 f_1 \mathrm{vol})^2 = 1. \tag{6.6}$$

By virtue of (2.12), (2.14) and (6.2), the last condition is equivalent to

$$m \equiv 1 \pmod 2 \quad \text{for signature } (p+q, 0), \tag{6.7}$$

$$m + p \equiv 0 \pmod 2 \quad \text{for signature } (p, q). \tag{6.8}$$

If either (6.7) or (6.8) is satisfied, then there are two lifts $\hat{\tau}_\pm$ given by

$$\hat{\tau}_\pm|_{H_0} = \hat{\tau}_0, \qquad \hat{\tau}_\pm(e_1 e_{p+1}, f_1 f_{q+1}) = \pm e_1 f_1 \mathrm{vol}. \tag{6.9}$$

According to theorem 1, the lifts $\hat{\tau}_+$ and $\hat{\tau}_-$ define two inequivalent spin structures on the quadric and every spin structure is equivalent to one of those two. This justifies the entries in table 1 corresponding to the first four lines.

(ii) Let $p = 1, q = 2m - 1$ and $m \geq 2$. The simply connected group

$$G = \mathbb{R} \times \mathrm{Spin}(2m) \tag{6.10}$$

acts transitively on $Q_{1,2m-1}$ by

$$(t,a)[(x,y)] = [(x \exp 2\pi\sqrt{-1}t, aya^{-1})],$$

where $(t,a) \in G, x \in \mathrm{U}(1)$, and $y \in \mathrm{S}_{2m-1}$. The connected component of the unit in the isotropy group $H$ of the point $[(1, f_{2m})]$ is

$$H_0 = \{0\} \times \mathrm{Spin}(2m-1).$$

The isotropy group $H$ is generated, as a subgroup of $G$, by $H_0$ and the element

$$(\tfrac{1}{2}, f_1 f_{2m}). \tag{6.11}$$

The linear isotropy representation is as in (i), viz.,

$$\tau : H \to \{1\} \times \mathrm{SO}(2m-1),$$

where

$$\tau(0,a) = (1, \rho(a)), \qquad \tau(\tfrac{1}{2}, f_1 f_{2m}) = (1, -F_1).$$

The restriction of $\tau$ to $H_0$ lifts to

$$\hat{\tau}_0 : H_0 \to \mathrm{Spin}(2m) \quad \text{or} \quad \mathrm{Spin}(1, 2m-1), \quad \hat{\tau}_0(0,a) = a.$$

The element $f_2 \cdots f_{2m-1} \in \mathrm{Spin}(2m)$ or $\mathrm{Spin}(1, 2m-1)$ covers $(1, -F_1)$. Irrespective of whether the quadric $Q_{1,2m-1}$ is given a metric of Euclidean or Lorentzian signature, there are, for every $m \geq 2$, two different lifts $\hat{\tau}_+$ and $\hat{\tau}_-$ of $\tau$; they are determined by

$$\hat{\tau}_\pm|_{H_0} = \hat{\tau}_0, \qquad \hat{\tau}_\pm(\tfrac{1}{2}, f_1 f_{2m}) = \pm f_2 \cdots f_{2m-1}. \tag{6.12}$$

There is no condition analogous to (6.6) because the square of the element (6.11), equal to $(1, -1)$, is not in $H_0$.

(iii) The rather well-known case of the torus $Q_{1,1}$ can be treated as follows. Make $G = \mathbb{R} \times \mathbb{R}$ act on $Q_{1,1} \cong (\mathrm{U}(1) \times \mathrm{U}(1))/\mathbb{Z}_2$ by

$$(s,t)[(x,y)] = [(x \exp 2\pi\sqrt{-1}s, y \exp 2\pi\sqrt{-1}t)].$$

The isotropy group of $[(1,1)]$,

$$H = \{(s,t) : \text{ either } (s,t) \in \mathbb{Z} \times \mathbb{Z} \text{ or } (s + \tfrac{1}{2}, t + \tfrac{1}{2}) \in \mathbb{Z} \times \mathbb{Z}\},$$

is generated, as a subgroup of $G$, by $H_0 = \{(0,0)\}$ and the elements $(\frac{1}{2}, \frac{1}{2})$, $(1,0)$ and $(0,1)$, subject to

$$(\tfrac{1}{2}, \tfrac{1}{2})^2 = (1,0) \cdot (0,1).$$

The representation $\tau$ is trivial and it has four different lifts to Spin(2) or Spin(1, 1), characterized by the four independent choices of signs,

$$\hat{\tau}(\tfrac{1}{2},\tfrac{1}{2}) = \pm 1, \qquad \hat{\tau}(1,0) = \pm 1. \tag{6.13}$$

The choice of positive signs in (6.13) corresponds to a "trivial" spin structure. Among the proper real quadrics there are three that are group manifolds and thus have trivial (product) spin structures. They are $Q_{1,1} \cong \mathrm{U}(1) \times \mathrm{U}(1)$, $Q_{1,3} \cong Q_{3,1} \cong \mathrm{U}(2)$ and $Q_{3,3} \cong \mathrm{SO}(4)$. The quadrics $Q_{1,7} \cong Q_{7,1}$, $Q_{7,3} \cong Q_{3,7}$ and $Q_{7,7}$ are parallelizable and, therefore, also have a trivial spin structure.

## 7. The topological conditions

In this section we recall the statement of the topological conditions on the existence of spin structures and apply them to the proper real quadrics. We use the results on the mod 2 cohomology groups of the quadrics given by Dieudonné [13] and compute the relevant Stiefel–Whitney classes [23].

Recall that, if $E \to M$ is a real vector bundle of fibre dimension $n$, then its $i$th Stiefel–Whitney class, $i = 0, 1, \ldots, n$, is an element of the $i$th cohomology group $H^i(M, \mathbb{Z}_2)$ such that the following axioms hold: (i) naturality, (ii) the Whitney product property, and (iii) the first Stiefel–Whitney class of the Möbius line bundle over the circle is non-zero. Denoting by $w_i(E)$ the $i$th class, and introducing the total Stiefel–Whitney class of $E \to M$,

$$w(E) = w_0(E) + w_1(E) + \cdots + w_n(E), \quad w_0(E) = 1,$$

one can write axiom (ii) as

$$w(E \oplus F) = w(E)w(F), \tag{7.1}$$

where $E \oplus F$ is the Whitney sum of the vector bundles $E \to M$ and $F \to M$. The multiplication on the right of (7.1) is the cup product in the cohomology algebra $\bigoplus_{i=0}^{n} H^i(M, \mathbb{Z}_2)$.

The vanishing of $w_1(E)$ is equivalent to the orientability of the vector bundle $E \to M$. The second Stiefel–Whitney class is related to the existence of spin structures; for our purposes it is convenient to formulate this relation in the following proposition, which is a corollary from a general theorem given by Karoubi [16]:

**Proposition 9.** *Let $M$ be an orientable pseudo-Riemannian manifold with a metric tensor $g$ of signature $(p, q)$ and let*

$$TM = T' \oplus T''$$

*be a decomposition of its tangent bundle into vector bundles such that $g|_{T'}$ $(g|_{T''})$ is positive (negative) definite. There is a Spin$(p,q)$-structure on $(M, g)$ iff*

$$w_2(T') + w_2(T'') = 0. \tag{7.2}$$

*Moreover, if this condition is satisfied, then inequivalent spin structures are in a bijective correspondence with the elements of* $H^1(M, \mathbb{Z}_2)$.

Note that orientability of the manifold $M$ is equivalent to

$$w_1(T') + w_1(T'') = 0. \tag{7.3}$$

It follows from (7.1) that

$$w_2(TM) = w_2(T') + w_2(T'') + w_1(T')w_1(T'')$$

and condition (7.2) is equivalent to

$$w_2(TM) + w_1(T')w_1(T'') = 0. \tag{7.4}$$

If $M$ is an orientable, proper Riemannian manifold, then $w_1(T') = w_1(TM) = 0$ and condition (7.4) reduces to

$$w_2(TM) = 0. \tag{7.5}$$

It is clear that to determine the existence of spin structures on the real quadric $Q_{p,q}$ ($p, q > 0$ and $p + q$ even) it suffices to compute the Stiefel–Whitney classes

$$w_1(p,q) = w_1(T'_{p,q}) = w_1(T''_{p,q}),$$

$$w'_2(p,q) = w_2(T'_{p,q}), \qquad w''_2(p,q) = w_2(T''_{p,q}),$$

where $T_{p,q} = T'_{p,q} \oplus T''_{p,q}$, as in proposition 3. According to proposition 9, the quadric $Q_{p,q}$ ($p + q$ even) has a

$$\text{Spin}(p+q)\text{-structure} \quad \text{iff } w'_2(p,q) + w''_2(p,q) + w_1(p,q)^2 = 0, \tag{7.6}$$

$$\text{Spin}(p,q)\text{-structure} \quad \text{iff } w'_2(p,q) + w''_2(p,q) = 0. \tag{7.7}$$

According to §24.39, prob. 10 of ref. [13], the cohomology group $H^i(Q_{p,q}, \mathbb{Z}_2)$ is

$$\begin{array}{rl} \mathbb{Z}_2 & \text{for } p < q,\ 0 \le i \le p,\ q \le i \le p+q, \\ \mathbb{Z}_2 & \text{for } p = q,\ 0 \le i \le p-1,\ p+1 \le i \le 2p, \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{for } p = q = i, \\ \{0\} & \text{for } p < i < q. \end{array}$$

Recall that, for $q$ odd, the quadric $Q_{1,q}$ is diffeomorphic to $S_1 \times S_q$ and, therefore, its Stiefel–Whitney classes $w_i$ vanish for $i > 0$. We need to consider further only the cases when $p$ and $q \ge 2$. Under this assumption, the group $H^1(Q_{p,q}, \mathbb{Z}_2)$ is $\mathbb{Z}_2$ generated by $\omega$, the class of the cocycle taking value 1 on the one-cycle defined by the map $c : [0, \pi] \to Q_{p,q}$, where

$$c(t) = [(e_1 \cos t + e_2 \sin t,\ f_1 \cos t + f_2 \sin t)]. \tag{7.8}$$

Moreover, the square of $\omega$—in the sense of the cup product—is a non-zero element of $H^2(Q_{p,q}, \mathbb{Z}_2)$ and a generator of the group except when $p = q = 2$.

To compute the first two Stiefel–Whitney classes of the vector bundle $T'_{p,q} \to Q_{p,q}$, we consider the bundle map

$$T'_{p,q} \to T\mathbb{RP}_p : [(x,u),(y,0)] \mapsto [(x,u)],$$

which covers the projection $\pi_p$ of $Q_{p,q}$ onto the real, $p$-dimensional projective space $\mathbb{RP}_p$, $\pi_p[(x,y)] = [x]$. From naturality we have

$$\pi_p^* w(T\mathbb{RP}_p) = w(T'_{p,q}).$$

According to theorem 4.5 in ref. [23], the total Stiefel–Whitney class of the real projective space is

$$w(T\mathbb{RP}_p) = (1+\tilde{\omega})^{p+1},$$

where $\tilde{\omega}$ is the generator of $H^1(\mathbb{RP}_p, \mathbb{Z}_2) = \mathbb{Z}_2$ characterized by the property of taking value 1 on the class of the one-cycle $\pi_p \circ c$, where $c$ is given by (7.8). This shows $\pi_p^* \tilde{\omega} = \omega$ and proves the formulae

$$\begin{aligned} w_1(p,q) &= (p+1)\omega, \\ w_2'(p,q) &= \tfrac{1}{2}p(p+1)\omega^2, \qquad w_2''(p,q) = \tfrac{1}{2}q(q+1)\omega^2, \end{aligned}$$

where $p, q \geq 2$ and the coefficients can be reduced mod 2. For $p + q = 2m$ we have the congruence

$$\tfrac{1}{2}p(p+1) + \tfrac{1}{2}q(q+1) \cong m + p \pmod 2.$$

Therefore, if $p$ and $q$ are odd and larger than 1, then $w_1(p,q) = 0$,

$$w_2'(p,q) + w_2''(p,q) = 0 \quad \text{iff } m \text{ is odd}, \tag{7.9}$$

there are two $\mathrm{Spin}(p+q)$-structures and two $\mathrm{Spin}(p,q)$-structures for $m$ odd, but none for $m$ even.

If $p$ and $q$ are even and $\geq 2$, then $w_1(p,q) = \omega \neq 0$,

$$w_2'(p,q) + w_2''(p,q) = \begin{cases} 0 & \text{for } m \text{ even}, \\ w_1(p,q)^2 \neq 0 & \text{for } m \text{ odd}. \end{cases} \tag{7.10}$$

There are two $\mathrm{Spin}(p+q)$-structures for $m$ odd and two $\mathrm{Spin}(p,q)$-structures for $m$ even, but not otherwise. These observations, together with earlier remarks about the case $p = 1$ and $q$ odd, give a complete topological justification of the results summarized in section 1 and described in section 6.

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